NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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TWO-DIMENSIONAL MOTION OF A GAS AT LARGE SUPERSONIC VELOCITIES

By S. V. Falkovich

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TWO-DIMENSIONAL MOTION OF A GAS AT LARGE SUPERSONIC VELOCITIES*

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A large number of papers have been devoted to the problem of integration of equations of two-dimensional steady nonvortical adiabatic motion of a gas. Most of these papers are based on the application of the hodograph method of S. A. Chaplygin in which the plane of the hodograph of the velocity is taken as the region of variation of the independent variables in the equations of motion; the equations become linear in this plane. The exact integration of these equations is, however, obtained in the form of infinite series containing hypergeometric functions. The obtaining of such solutions and their investigation involve extensive computations. As a result, methods have been developed for the approximate integration of the equations of motion first transformed to a linear form. S. A. Chaplygin in reference 1 first pointed out such an approximate method applicable to flows in which the Mach number does not exceed 0.4.

S. A. Christianovich (reference 2), in solving the problem of the flow with circulation about a wing in a supersonic stream, gave as a first approximation a generalization of the method of Chaplygin to the case where the region of variation of the velocity in the hodograph plane lies within a sufficiently narrow ring entirely inside the circle $W < a_*$. At the same time and independently an analogous method was proposed by Tsien and von Kármán (references 3 and 4). These methods are not applicable for Mach numbers near unity. In the papers by F. I. Frankl (reference 5) and S. V. Falkovich (reference 6), approximate equations of motion are given suitable for the investigation of flows in passing through the velocity of sound. In his recently published paper, S. A. Christianovich (reference 7) showed that for Mach numbers $1.05 < M < 3.5$ the equations for the stream function and the velocity potential may approximately be replaced by the equations of Darboux with integral coefficients and hence the general integral obtained in finite form.

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The equation of motion of a gas is investigated herein for large supersonic velocities and a method is shown for the approximate integration, which gives sufficient accuracy for Mach numbers greater than 4.

1. Fundamental equation and its transformation. - The equation determining the velocity potential of the two-dimensional irrotational adiabatic motion of a gas has, as known, the form

$$\left(a^2 - u^2\right) \frac{\partial^2 \varphi}{\partial x^2} - 2uv \frac{\partial^2 \varphi}{\partial x \, \partial y} + \left(a^2 - v^2\right) \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{1.1}$$

To this equation the transformation of Legendre is applied. As the independent variables in place of x and y are taken $\partial\varphi/\partial x = u$ and $\partial\varphi/\partial y = v$ and in place of the velocity potential $\varphi(x,y)$, the Legendre potential is introduced.

$$\Phi(u,v) = ux + vy - \varphi(x,y) \tag{1.2}$$

Equation (1.1) becomes

$$\left(a^2 - v^2\right) \frac{\partial^2 \Phi}{\partial u^2} + 2uv \frac{\partial^2 \Phi}{\partial u \, \partial v} + \left(a^2 - u^2\right) \frac{\partial^2 \Phi}{\partial v^2} = 0 \tag{1.3}$$

If the function $\Phi(u,v)$ is determined, then by differentiation of equation (1.2), the coordinates x and y of the corresponding point in the flow plane are determined.

$$x = \partial\Phi/\partial u \qquad y = \partial\Phi/\partial v \tag{1.4}$$

Pass in the hodograph plane u,v to polar coordinates and set $u = W \cos \theta$ and $v = W \sin \theta$; equations (1.3) and (1.4) then become

$$\frac{a^2 - W^2}{a^2 W^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial W^2} + \frac{a^2 - W^2}{a^2 W} \frac{\partial \Phi}{\partial W} = 0 \tag{1.5}$$

$$x = \cos \theta \frac{\partial \Phi}{\partial W} - \frac{\sin \theta}{W} \frac{\partial \Phi}{\partial \theta} \qquad y = \sin \theta \frac{\partial \Phi}{\partial W} + \frac{\cos \theta}{W} \frac{\partial \Phi}{\partial \theta} \tag{1.6}$$

If only supersonic velocities are considered, it is more convenient in place of the independent variable W to introduce in equation (1.5) the new variable

$$z = 1/\sqrt{M^2 - 1} \tag{1.7}$$

Thus z will be small for large Mach numbers and will increase indefinitely as $M \rightarrow 1$.

In order to carry out this transformation, from the equation of Bernoulli the expression of W is found in terms of z. Thus

$$\frac{W^2}{2} + \frac{a^2}{\kappa - 1} = \frac{h^2 a_*}{2} \qquad \left(h^2 = \frac{\kappa+1}{\kappa-1} \approx 6\right)$$

or

$$\frac{1}{2} + \frac{1}{(\kappa-1)\ M^2} = \frac{h^2 {a_*}^2}{2W^2}$$

If M^2 is replaced by z, from equation (1.7)

$$W^2 = \frac{h^2 {a_*}^2 (1+z^2)}{1+h^2 z^2} \tag{1.8}$$

By making use of this expression, pass in equations (1.5) and (1.6) from the variable W to the variable z; then

$$\frac{\partial^2 \Phi}{\partial z^2} - \frac{(h^2-1)^2}{(1+z^2)^2\ (1+h^2z^2)^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{3h^2z^4 + 2h^2z^2 + h^2 - 2}{(1+z^2)\ (1+h^2z^2)z} \frac{\partial \Phi}{\partial z} = 0 \tag{1.9}$$

$$x = -\frac{1}{W}\left[\frac{(1+z^2)\ (1+h^2z^2)}{(h^2-1)z} \cos\theta \frac{\partial \Phi}{\partial z} + \sin\theta \frac{\partial \Phi}{\partial \theta}\right]$$

$$y = -\frac{1}{W}\left[\frac{(1+z^2)\ (1+h^2z^2)}{(h^2-1)z} \sin\theta \frac{\partial \Phi}{\partial z} - \cos\theta \frac{\partial \Phi}{\partial \theta}\right] \tag{1.10}$$

The characteristics of equation (1.9) may be taken in the form

$$\lambda = \theta - \int_0^z \frac{(h^2-1)dz}{(1+z^2)\ (1+h^2z^2)} \qquad \mu = \theta + \int_0^z \frac{(h-1)dz}{(1+z^2)\ (1+h^2z^2)}$$

The line of maximum velocity $z = 0$ in the hodograph plane will correspond in the plane of the characteristics $\lambda\mu$ to the line $\mu - \lambda = 0$. If the integration is carried out,

$$\lambda = \theta - (h \operatorname{arc\,tg} hz - \operatorname{arc\,tg} z) \qquad \mu = \theta + (h \operatorname{arc\,tg} hz - \operatorname{arc\,tg} z) \tag{1.11}$$

2. Investigation of equation (1.9). - If equation (1.9) is referred to the characteristic coordinates $\lambda\mu$,

$$\frac{\partial^2\Phi}{\partial\lambda\,\partial\mu} - \frac{h^2 - 2 - 2z^2 - h^2z^4}{4(h^2 - 1)z}\left(\frac{\partial\Phi}{\partial\mu} - \frac{\partial\Phi}{\partial\lambda}\right) = 0 \tag{2.1}$$

From equations (1.11),

$$\mu - \lambda = 2(h \operatorname{arc\,tg} hz - \operatorname{arc\,tg} z) \tag{2.2}$$

From equation (2.2) it follows that the coefficient of the equation (2.1)

$$L(\mu - \lambda) = \frac{h^2 - 2 - 2z^2 - h^2z^4}{4(h^2 - 1)z} \tag{2.3}$$

is a function of the difference $\mu - \lambda$. Equations (2.2) and (2.3) give this dependence in parametric form.

From equation (2.3) it follows that the function $L(\mu - \lambda)$ is negative for $z^2 > (h^2 - 2)/h^2$ and positive for $z^2 < (h^2 - 2)/h^2$. For $z^2 = (h^2 - 2)/h^2$ the function $L(\mu - \lambda)$ becomes zero. According to equation (1.7) this function corresponds to the Mach number

$$M = 2/\sqrt{3-\kappa} = 1.565$$

S. A. Christianovich showed (reference 8) that for a given value of the Mach number there is a change in the direction of curvature of the characteristics in the flow plane. The graph of the function $L(\mu - \lambda)$ is shown in figure 1.

If the right side of equation (2.2) is expanded in a series in powers of z,

$$\frac{\mu - \lambda}{2} = (h^2 - 1) z - \frac{h^4 - 1}{3} z^3 + \ldots \tag{2.4}$$

from which

$$z = \frac{\mu - \lambda}{2(h^2 - 1)} + \frac{h^2 + 1}{24(h^2 - 1)^3} (\mu - \lambda)^3 + \ldots$$

If this series is substituted in equation (2.3),

$$L(\mu - \lambda) = \frac{h^2 - 2}{2(\mu - \lambda)} - \frac{(h^2 + 1)(h^2 - 2) + 6}{24(h^2 - 1)^2} (\mu - \lambda) + \ldots$$

Set $h^2 = 6$. Then

$$L(\mu - \lambda) = \frac{2}{\mu - \lambda} - 0.057 (\mu - \lambda) + \ldots$$

For $M > 4$, assume

$$L(\mu - \lambda) \approx \frac{2}{\mu - \lambda}$$

Equation (2.1) then becomes

$$\frac{\partial^2 \Phi}{\partial \lambda \, \partial \mu} - \frac{2}{\lambda - \mu} \left(\frac{\partial \Phi}{\partial \lambda} - \frac{\partial \Phi}{\partial \mu} \right) = 0 \tag{2.5}$$

Equation (2.5) is the equation of Darboux with integral coefficient. The general integral of this equation has the form

$$\Phi(\lambda, \mu) = \frac{\partial^2}{\partial \lambda \, \partial \mu} \frac{X(\lambda) - Y(\mu)}{\lambda - \mu} = \frac{X'(\lambda) + Y'(\mu)}{(\lambda - \mu)^2} - 2 \frac{X(\lambda) - Y(\mu)}{(\lambda - \mu)^3} \tag{2.6}$$

where $X(\lambda)$ and $Y(\mu)$ are arbitrary functions of their arguments. The expression (2.6) is the asymptotic integral of the exact equation (2.1) for $z \to 0$, that is, for $M \to \infty$.

Equations (1.10) for the coordinates x and y, after transformation to the characteristics $\lambda\,\mu$, becomes on the basis of equation (1.11)

$$\left.\begin{aligned} x &= -\frac{1}{W}\left[\left(\frac{\partial\Phi}{\partial\mu}+\frac{\partial\Phi}{\partial\lambda}\right)\sin\frac{\lambda+\mu}{2}+\frac{1}{z}\left(\frac{\partial\Phi}{\partial\mu}-\frac{\partial\Phi}{\partial\lambda}\right)\cos\frac{\lambda+\mu}{2}\right] \\ y &= +\frac{1}{W}\left[\left(\frac{\partial\Phi}{\partial\mu}+\frac{\partial\Phi}{\partial\lambda}\right)\cos\frac{\lambda+\mu}{2}-\frac{1}{z}\left(\frac{\partial\Phi}{\partial\mu}-\frac{\partial\Phi}{\partial\lambda}\right)\sin\frac{\lambda+\mu}{2}\right] \end{aligned}\right\} \quad (2.7)$$

From equation (2.2) for small values of z,

$$z = \frac{\mu-\lambda}{2(h^2-1)} = \frac{\mu-\lambda}{10}$$

If this value of z is substituted in equation (2.7) and the arbitrary functions $\Phi(\lambda,\mu)$ eliminated from equation (2.7), with the aid of equation (2.6) the final solution is obtained.

$$x = -\frac{1}{W}\left[\left(\frac{X''+Y''}{(\lambda-\mu)^2}-2\,\frac{X'-Y'}{(\lambda-\mu)^3}\right)\sin\frac{\lambda+\mu}{2}-\right.$$

$$\left.10\left(\frac{X''-Y''}{(\lambda-\mu)^3}-6\,\frac{X'+Y'}{(\lambda-\mu)^4}+12\,\frac{X-Y}{(\lambda-\mu)^5}\right)\cos\frac{\lambda+\mu}{2}\right]$$

$$y = +\frac{1}{W}\left[\left(\frac{X''+Y''}{(\lambda-\mu)^2}-2\,\frac{X'-Y'}{(\lambda-\mu)^3}\right)\cos\frac{\lambda+\mu}{2}+\right.$$

$$\left.10\left(\frac{X''-Y''}{(\lambda-\mu)^3}-6\,\frac{X'+Y'}{(\lambda-\mu)^4}+12\,\frac{X-Y}{(\lambda-\mu)^5}\right)\sin\frac{\lambda+\mu}{2}\right]$$

Consider the equations for the velocity potential $\varphi(W,\theta)$ and the stream function $\psi(W,\theta)$:

$$\frac{\partial\varphi}{\partial\theta} = \frac{\rho_0}{\rho}\,W\,\frac{\partial\psi}{\partial W} \qquad \frac{\partial\varphi}{\partial W} = \frac{\rho_0}{\rho}\,(M^2-1)\,\frac{1}{W}\,\frac{\partial\psi}{\partial\theta}$$

If the variable z is introduced in place of the modulus of the velocity W according to equation (1.7) and equation (1.8) used, these equations are transformed to the form

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= -\left(1 + h^2 z^2\right)^{\frac{\kappa}{\kappa-1}} \left(h^2 - 1\right)^{\frac{\kappa}{1-\kappa}} \left(1 + z^2\right) z^{\frac{\kappa+1}{1-\kappa}} \frac{z\psi}{\partial z} \\ \frac{\partial\varphi}{\partial z} &= -\left(1 + h^2 z^1\right)^{\frac{2-\kappa}{\kappa+1}} \left(1 + z^2\right)^{-1} \left(h^2 - 1\right)^{-1} z^{\frac{\kappa+1}{1-\kappa}} \frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \qquad (2.8)$$

For small z, equations (2.8) may be replaced approximately by the equations

$$\frac{\partial\varphi}{\partial\theta} = -\left(h^2 - 1\right)^{\frac{\kappa}{1-\kappa}} z^{\frac{\kappa+1}{1-\kappa}} \frac{\partial\psi}{\partial z}$$

$$\frac{\partial\varphi}{\partial z} = -\left(h^2 - 1\right)^{\frac{2-\kappa}{\kappa-1}} z^{\frac{\kappa+1}{1-\kappa}} \frac{\partial\psi}{\partial z}$$

If the velocity potential φ is eliminated, the equation for the stream function ψ is

$$\left(h^2 - 1\right)^2 \frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial z^2} + \frac{h^2}{z} \frac{\partial\psi}{\partial z} = 0 \qquad (2.9)$$

the characteristics of which have the form

$$\lambda = \theta - \left(h^2 - 1\right)z \qquad \mu = \theta + \left(h^2 - 1\right)z$$

If equation (2.9) is referred to the characteristics, it is transformed into the form

$$\frac{\partial^3\psi}{\partial\lambda\,\partial\mu} + \frac{h^2}{2(\lambda - \mu)}\left(\frac{\partial\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\mu}\right) = 0$$

Setting $h^2 = 6$ gives

$$\frac{\partial^2\psi}{\partial\lambda\,\partial\mu} + \frac{3}{\lambda - \mu}\left(\frac{\partial\psi}{\partial\lambda} - \frac{\partial\psi}{\partial\mu}\right) = 0 \qquad (2.10)$$

The equation of Darboux with integral coefficient, which is analogous to equation (2.5), is integrated in finite form.

3. Criterions of similarity. - By examining equation (1.9) for Mach numbers near 1 $(z \rightarrow \infty)$ and also for large Mach numbers $(z \rightarrow 0)$, certain criterions of similarity can be established that may be useful in evaluating experimental data obtained in wind-tunnel tests. For large values of z, equation (1.9) can be replaced by the approximate equation

$$\frac{\partial^2 \Phi}{\partial z^2} - \frac{(h^2 - 1)^2}{h^4 z^8} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{3}{z} \frac{\partial \Phi}{\partial z} = 0 \tag{3.1}$$

and for small values of z by the approximate equation

$$\frac{\partial^2 \Phi}{\partial z^2} - (h^2 - 1)^2 \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{h^2 - 2}{z} \frac{\partial \Phi}{\partial z} = 0 \tag{3.2}$$

A thin slightly cambered airfoil at small angle of attack is now considered in a plane-parallel nonvertical gas flow with Mach number M_0 at a large distance from the airfoil. The profile chord is denoted by l and its maximum thickness is denoted by δ. If it is assumed that M_0 is near unity, in equation (3.1)

$$\theta = \theta^* \delta / l \qquad z = z^* z_0 \qquad \left(z_0 = 1 / \sqrt{M_0^{\,2} - 1}\right)$$

and equation (3.1) becomes

$$K_1^{\,2} \frac{\partial^2 \Phi}{\partial z^{*2}} - \frac{h^2 - 1}{h^4 z^{*8}} \frac{\partial^2 \Phi}{\partial \theta^{*2}} + \frac{3}{z^*} \frac{\partial \Phi}{\partial z^*} = 0 \tag{3.3}$$

where

$$K_1 = \frac{\delta}{l} \sqrt{(M^2 - 1)^3} \tag{3.4}$$

The number K_1 may be called the criterion of similarity in the sense that two flows having different Mach numbers about two airfoils of different thicknesses δ and different chords l but for which the values of K_1 are the same will be determined by the same "nondimensional" equation (3.3); hence, for the drag coefficient C_x

$$C_x = f(K_1)$$

In an entirely analogous manner starting from equation (3.2), a second criterion of similarity valid for large Mach numbers is

$$K_2 = \frac{\delta}{l} \sqrt{M_0^{\ 2} - 1} \approx \frac{\delta}{l} M_0$$

This criterion of similarity has recently been obtained by Tsien by a different considerably more complicated method.

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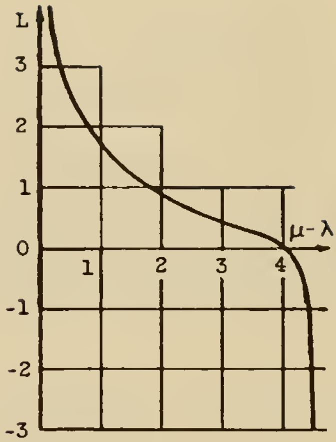

Figure 1.

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Two-Dimensional Motion of a Gas at Large Supersonic Velocities.

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Abstract

The equation of motion of a gas is investigated for large supersonic velocities and a method is shown for the approximate integration, which gives sufficient accuracy for Mach numbers greater than 4.